Towards a Supersymmetric Doubled Worldsheet Formalism

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Introduction

- Non-geometric compactifications.
- The recent results on double field theory of Zwiebach, Hohm, Siegel, Hull, ... point to the existence of a "stringy" generalized geometrical framework.
- Many expressions appearing in double field theory – even for the non-supersymmetric case – are reminiscent of generalized Kähler and Calabi-Yau geometry found in $N = (2, 2)$, $d = 2$ non-linear σ-models...
- Try to get a handle on this – at least for the NSR sector – by means of a manifest T-dual invariant, $N = (2, 2)$ worldsheet description (both classical and quantum, inspired by Tseytlin, Hull, ...).

Introduction: T-duality

- Consider a space-time with one coordinate y compactified on a circle with radius R: $y = y + 2\pi R$. Denote the other coordinates by x. Take a massless scalar field $\varphi(x, y)$:

$$\varphi(x,y) = \sum_{n\in\mathbb{Z}} \varphi_n(x) e^{i\,n\,y/R}$$

$\varphi_n(x)$ has mass M:

$$M^2 = \frac{n^2}{R^2}$$

When $R \to 0$ we end up with a theory in $d - 1$ dimensions.

Introduction: T-duality

- The situation changes when we consider a closed string instead of a point particle. A string can wind around a compact direction. For a string with winding number m, the mass formula changes to (α' is the length squared of the string),

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \cdots$$

- T-duality:

$$\begin{aligned} m &\leftrightarrow n \\ R &\rightarrow \frac{\alpha'}{R} \end{aligned}$$

- This can be generalized to non-trivial backgrounds having a number of (abelian) isometries.

Buscher rules

- In the RNS formulation, bosonic strings propagating on a manifold M are characterized by a metric g, a closed 3-form $T = db$ and a dilaton Φ. Assume now a number p of (abelian) isometries and go to adapted coordinates: x^a, y^i ($i \in \{1, \cdots p\}$, $a \in \{1, \cdots d-p\}$) such that neither g, b nor Φ depend on the y-coordinates (we always assume that the background fields do depend on the x-coordinates). Isometry:

$$y^i \to y^i + \xi^i.$$

- Buscher procedure: promote the isometry to a gauge symmetry & using Lagrange multipliers impose that the gauge fields are pure gauge.
- Integrate over the Lagrange multipliers $\Rightarrow$ original model, integrate over the gauge fields $\Rightarrow$ T-dual model!

Buscher rules

- Introduce $e \equiv g + b$. Original model: $E_{ij} \equiv e_{ij}$, $M_{ib} \equiv e_{ib}$, $N_{aj} \equiv e_{aj}$ and $F_{ab} \equiv e_{ab}$. Dual model:

$$\begin{aligned} \tilde{E} &= E^{-1} \\ \tilde{N} &= E^{-1} M \\ \tilde{N} &= -N E^{-1} \\ \tilde{F} &= F - N E^{-1} M \end{aligned}$$

  The dilaton transforms as well.
- To simplify the expressions we assume $N = M = 0$.

Buscher rules

- T-duality transformations form an $O(p, p, \mathbb{Z})$ group (includes large diffeomorphisms and constant b-shifts as well). Introduce η and $G \in O(p, p, \mathbb{Z})$,

$$\eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad G^T \eta G = \eta\,.$$

- $E_{ij} = g_{ij} + b_{ij}$ transforms non-linearly under $O(p, p, \mathbb{Z})$:

$$\tilde{E} = (A\,E + B)(C\,E + D)^{-1}\,.$$

Buscher rules

- Introduce $\mathcal{H}$,

$$\mathcal{H} = \begin{pmatrix} g - bg^{-1}b & -b\,g^{-1} \\ g^{-1}b & g^{-1} \end{pmatrix},$$

- $\mathcal{H}$ transforms linearly under $O(p, p, \mathbb{Z})$:

$$\tilde{\mathcal{H}} = G^T \mathcal{H} G\,.$$

- Note that $\mathcal{S} \equiv \eta\mathcal{H}$ satisfies $\mathcal{S}^2 = \mathbf{1}$.
- Doubled formalism (Tseytlin, Hull, ...): make $O(p, p, \mathbb{Z})$ invariance manifest.

Doubled formalism

- Double the coordinates: y^i and $\tilde{y}_i$. Introduce $\mathbb{Y}$,

$$\mathbb{Y} = \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} ,$$

- The doubled worldsheet lagrangian is now,

$$\mathcal{L} = \frac{1}{2} \partial_{\ddagger} \mathbb{Y}^T \mathcal{H} \partial_{=} \mathbb{Y} + \mathcal{L}(x) ,$$

  where we used worldsheet light-cone coordinates: $\sigma^{\ddagger} = \tau + \sigma$ and $\sigma^{=} = \tau - \sigma$.

- In order to be equivalent with the original theory, this has to be supplemented with p constraints.

Doubled formalism

- Use the almost product structure $\mathcal{S} = \eta\mathcal{H}$ to introduce the projection operators $\mathbb{P}_\pm$:

$$\mathbb{P}_\pm \equiv \frac{1}{2}(\mathbf{1} \pm \mathcal{S}),$$

and the constraints are given by,

$$\mathbb{P}_+ \partial_= \mathbb{Y} = \mathbb{P}_- \partial_{\mp} \mathbb{Y} = 0 .$$

In a more recognizable form they can be rewritten as,

$$\partial_{\mp}\tilde{y} = (g - b)\partial_{\mp} y \quad \partial_=\tilde{y} = -(g + b)\partial_= y .$$

- Can this be rewritten in a first order form? Chiral boson!

Digression: chiral bosons in $d = 2$

- Floreanini-Jackiw:

$$\mathcal{L} = \partial_\tau \phi \partial_\sigma \phi - \partial_\sigma \phi \partial_\sigma \phi \, .$$

Eq. of motion:

$$\partial_\sigma \partial_= \phi = 0 \rightarrow \partial_= \phi = g(\tau) \, ,$$

where $g(\tau)$ can be put to zero by a gauge choice.

$$\Rightarrow \partial_= \phi = 0 \leftrightarrow \text{ chiral boson.}$$

- Floreanini-Jackiw is not Lorentz invariant. Lorentz invariant formulation? Yes: Siegel & PST.

Digression: chiral bosons in $d = 2$

- Siegel:

$$\mathcal{L} = \partial_{\ddagger}\phi\,\partial_{=}\phi - h_{\ddagger\ddagger}\partial_{=}\phi\,\partial_{=}\phi\,.$$

Gauge invariance (chiral gravity):

$$\begin{aligned} \delta\phi &= \varepsilon^{=}\partial_{=}\phi\,, \\ \delta h_{\ddagger\ddagger} &= \partial_{\ddagger}\varepsilon^{=} + \varepsilon^{=}\partial_{=}h_{\ddagger\ddagger} - \partial_{=}\varepsilon^{=}\,h_{\ddagger\ddagger} \end{aligned}$$

Make gauge choice $h_{\ddagger\ddagger} = 1 \Rightarrow$ Floreanini-Jackiw.

Digression: chiral bosons in $d = 2$

- PST: Introduce a Beltrami like parameterization for $h_{\ddagger\ddagger}$:

$$h_{\ddagger\ddagger} = \frac{\partial_{\ddagger} f}{\partial_{=} f} \,.$$

$$\mathcal{L} = \partial_{\ddagger}\phi\,\partial_{=}\phi - \frac{\partial_{\ddagger} f}{\partial_{=} f}\,\partial_{=}\phi\,\partial_{=}\phi \,,$$

PST gauge invariance:

$$\begin{aligned} \delta f &= \varepsilon^{=}\partial_{=} f \quad \rightarrow \quad \delta f = \xi \,, \\ \delta\phi &= \varepsilon^{=}\partial_{=}\phi \quad \rightarrow \quad \delta\phi = \xi\,\frac{\partial_{=}\phi}{\partial_{=} f} \,. \end{aligned}$$

- Can we do this for the doubled formalism?

First order doubled formalism

- Introduce $h_{\mp\mp}$ and $h_{==}$ transforming as,

$$\begin{aligned}\delta h_{\mp\mp} &= \partial_{\mp}\varepsilon^{=} + \varepsilon^{=}\partial_{=}h_{\mp\mp} - \partial_{=}\varepsilon^{=}\, h_{\mp\mp},\\ \delta h_{==} &= \partial_{=}\varepsilon^{\mp} + \varepsilon^{\mp}\partial_{\mp}h_{==} - \partial_{\mp}\varepsilon^{\mp} h_{==}\,,\end{aligned}$$

  and repeat Siegel's construction.
- Works only when $\mathcal{H}$ is constant. Instead one requires,

$$h_{\mp\mp}h_{==} = 1 \;\Rightarrow\; \varepsilon^{=} = \varepsilon^{\mp} h_{\mp\mp}.$$

First order doubled formalism

- This immediately gives the doubled formalism in a PST like formulation:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_{\ddagger}\mathbb{Y}^{T}\mathcal{H}\,\partial_{=}\mathbb{Y} - \frac{1}{2}\frac{\partial_{\ddagger}f}{\partial_{=}f}\,\partial_{=}\mathbb{Y}\eta\,\mathbb{P}_{+}\partial_{=}\mathbb{Y} \\ &\quad + \frac{1}{2}\frac{\partial_{=}f}{\partial_{\ddagger}f}\,\partial_{\ddagger}\mathbb{Y}\eta\,\mathbb{P}_{-}\partial_{\ddagger}\mathbb{Y} + \mathcal{L}(x)\end{aligned}$$

- Gauge invariance:

$$\begin{aligned}\delta\mathbb{Y} &= \frac{\xi}{\partial_{=}f}\,\mathbb{P}_{+}\partial_{=}\mathbb{Y} + \frac{\xi}{\partial_{\ddagger}f}\,\mathbb{P}_{-}\partial_{\ddagger}\mathbb{Y}\,, \\ \delta f &= \xi\,.\end{aligned}$$

First order doubled formalism

- Additional gauge invariance:

$$\begin{aligned} \delta\mathbb{Y} &= \zeta(f)\,, \\ \delta f &= 0\,. \end{aligned}$$

- Floreanini-Jackiw like formulation by making the gauge choice $\partial_\sigma f = 0$:

$$\mathcal{L} = \frac{1}{4}\left(\partial_\sigma\mathbb{Y}^T\eta\partial_\tau\mathbb{Y} - \partial_\sigma\mathbb{Y}^T\mathcal{H}\partial_\sigma\mathbb{Y}\right) + \mathcal{L}(x)\,,$$

  which was the starting point of numerous investigations.
- Can we repeat this in $N = (2, 2)$ superspace? Turns out to be very hard – surprisingly enough even going to $N = (1, 1)$ is very tough...

$N=(2,2)$ superspace

- Coordinates: $\sigma^{\ddagger}$, $\sigma^{=}$, θ^{+}, θ^{-}, $\hat{\theta}^{+}$, $\hat{\theta}^{-}$ and we introduce D_+, D_-, $\hat{D}_+$ and $\hat{D}_-$ satisfying,

$$D_+^2=\hat{D}_+^2=-\frac{i}{2}\,\partial_{\ddagger}, \qquad D_-^2=\hat{D}_-^2=-\frac{i}{2}\,\partial_{=}\,.$$

- Action:

$$\mathcal{S}=\int d^2\sigma\, d^2\theta\, d^2\hat{\theta}\,\mathcal{V}(X).$$

- $\mathcal{V}$ can only be some function of the scalar superfields $\Rightarrow$ constraints needed! In $d=2$, $N=(2,2)$ there are numerous types of superfields. Here: only superfields satisfying constraints linear in the derivatives. Sufficient to describe all $N=(2,2)$ non-linear σ-models (Lindström, Roček, von Unge, Zabzine '05) (AS, Troost '96).

$N = (2, 2)$ superfields

- Constrain both chiralities:

$$\hat{D}_+ X^a = J^a_{+\,b}(X) D_+ X^b \,, \qquad \hat{D}_- X^a = J^a_{-\,b}(X) D_- X^b \,.$$

- Integrability conditions $\Rightarrow$ J_+ and J_- are commuting complex structures which can be simultaneously diagonalized.

$N = (2,2)$ superfields

- Chiral superfields z and $\bar{z} = z^\dagger$:

$$\hat{D}_\pm z = +i\, D_\pm z\,, \qquad \hat{D}_\pm \bar{z} = -i\, D_\pm \bar{z}\,.$$

- Twisted chiral superfields (Gates, Hull, Roček, '84) w and $\bar{w} = w^\dagger$:

$$\hat{D}_\pm w = \pm i\, D_\pm w\,, \qquad \hat{D}_\pm \bar{w} = \mp i\, D_\pm \bar{w}\,.$$

$N=(2,2)$ superfields

- Constrain only one chirality: Semi-chiral superfields (Buscher, Lindström, Roček, '88) l, r, $\bar{l}=l^\dagger$ and $\bar{r}=r^\dagger$:

$$\hat{D}_+ l = +i\, D_+ l\,, \quad \hat{D}_- r = +i\, D_- r\,,$$
$$\hat{D}_+ \bar{l} = -i\, D_+ l\,, \quad \hat{D}_- \bar{r} = -i\, D_- r\,.$$

- All $N=(2,2)$ non-linear σ-models can be described in terms of chiral, twisted chiral and semi-chiral superfields. (Lindström, Roček, von Unge, Zabzine '07; AS, J. Troost, '96)
- For chiral and twisted superfields the constraints eliminate $3/4$ of the components while for semi-chiral superfields only half are eliminated (the rest are $N=(1,1)$ auxiliary superfields).

T-duality in $N = (2,2)$ superspace

T-duality in $N = (2,2)$ superspace changes the nature of the superfields:

- Chiral $\leftrightarrow$ twisted chiral (Gates, Hull, Roček, '84)

$$V(w + \bar{w}, \cdots) \leftrightarrow \tilde{V}(z + \bar{z}, \cdots)$$

- Chiral + twisted chiral $\leftrightarrow$ semi-chiral (Grisaru, Massar, AS, Troost, '98)

$$V(z + \bar{z}, w + \bar{w}, i(z - \bar{z} - w + \bar{w}), \cdots) \leftrightarrow \tilde{V}(l + \bar{l}, r + \bar{r}, i(l - \bar{l} - r + \bar{r}), \cdots)$$

An example: $SU(2) \times U(1) = S^3 \times S^1$

- Parameterization:

$$g = e^{i\rho} \begin{pmatrix} \cos\psi\, e^{-i\varphi_1} & \sin\psi\, e^{i\varphi_2} \\ -\sin\psi\, e^{-i\varphi_2} & \cos\psi\, e^{i\varphi_1} \end{pmatrix},$$

  and

$$\varphi_1,\, \varphi_2,\, \rho \in \mathbb{R} \,\mathrm{mod}\, 2\pi \text{ and } \psi \in [0, \pi/2]$$

- Can be described in terms of a chiral z and twisted chiral w superfield (Roček, Schoutens, AS '91)

$$z = i\,\rho + \varphi_2 - i \ln \sin\psi\,, \qquad w = i\,\rho + \varphi_1 - i \ln \cos\psi\,.$$

- Generalized Kähler potential:

$$\mathcal{V} = \int^{i(z-\bar{z}-w+\bar{w})} dq\, \ln(1+e^q) - \frac{1}{2}(w+\bar{w})^2\,.$$

An example: $SU(2) \times U(1) = S^3 \times S^1$

- The isometry $z \to z + i\,\xi$, $w \to w + i\,\xi$, corresponds to moving along the S^1 of $S^3 \times S^1$. Dualizing along this isometry sends S^1 to S^1 and provides an alternative description of the σ-model in terms of a semi-chiral multiplet. (Troost, AS '96) (AS, Staessens, Terryn '11) (Roček, Lindström '11)

Outlook

- Can we write down a manifest T-dual invariant formulation in $N = (2, 2)$ superspace?
- Requires more than just doubling the coordinates on which the isometries act. Full superfields are doubled (over doubling). E.g. $V(z + \bar{z}, \cdots) \leftrightarrow \tilde{V}(w + \bar{w}, \cdots)$, doubled formulation will contain both z and w.
- Full answer not known yet but a simple example shows already some of the main features.

A simple example

- Chiral field z, twisted chiral field w.
- Original potential,

$$V(z+\bar{z}) = \int^{z+\bar{z}} dq\, F(q)\,.$$

- T-dual potential:

$$\tilde{V}(w+\bar{w}) = -\int^{w+\bar{w}} d\tilde{q}\, F^{-1}(\tilde{q})\,.$$

A simple example

- Doubled potential $\mathbb{V}$:

$$\mathbb{V} = \frac{1}{2} V(z+\bar{z}) + \frac{1}{2} \tilde{V}(w+\bar{w}),$$

  and constraint,

$$w + \bar{w} = F(z+\bar{z})\,.$$

- The constraint,

$$w + \bar{w} = F(z+\bar{z})\,.$$

  eliminates the “over doubled coordinates” and implies Hull’s constraints, act on it with $\hat{D}_+$ and $\hat{D}_-$:

$$D_\pm(w-\bar{w}) = \pm F'(z+\bar{z})\, D_\pm(z-\bar{z})\,.$$

Open ending

- The first order PST like formulation of these models is hard and currently under construction (AS, Thompson). Even the Floreanini-Jackiw formulation in $N = (1, 1)$ superspace is not known...
- Once this is done the 1-loop β-functions can be computed and the resulting quantum geometry can be compared to the geometry being developed by Zwiebach, Hohm and collaborators.
- To be continued...